THE SCOTS COLLEGE

YEAR 12
MATHEMATICS
TRIAL

AUGUST 2005

WEIGHTING 40%

TIME ALLOWED: **3 HOURS** (PLUS 5 MINUTES READING TIME)

INSTRUCTIONS:

- ATTEMPT ALL QUESTIONS
- WRITE IN BLUE OR BLACK INK
- ALL QUESTIONS ARE OF EQUAL VALUE
- ANSWER EACH SECTION IN A SEPARATE BOOKLET
- APPROVED NON-PROGRAMMABLE CALCULATORS MAY BE USED
- A TABLE OF STANDARD INTEGRALS IS PROVIDED

Question 1 (12 Marks) Use a SEPARATE writing booklet **Marks**

a Convert a standard speed limit of 60 km/h to m/s correct to 2 decimal places. **2**

b Express $0.3\dot{1}\dot{5}$ as a fraction in its lowest terms. **2**

c Solve $\frac{x-1}{3}-\frac{x-2}{5}=3$ **2**

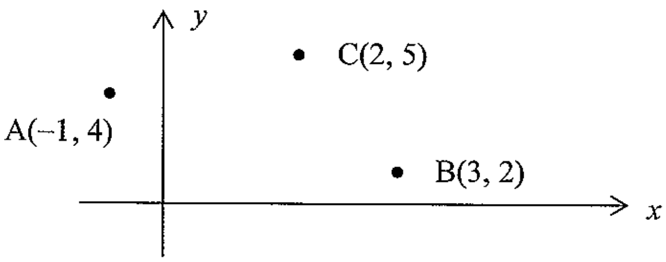
d Leaving your answer in exact form, simplify $\frac{1}{2\sqrt{3}-1}+\frac{1}{2\sqrt{3}+1}$ **2**

e \$5000 is left for 3 years in a savings account earning 4% pa. What would be the final value if the interest was compounded monthly? **2**

f Find the values of x for which $|x+3|<4$ **2**

Question 2 (12 Marks) Use a SEPARATE writing booklet.

a



(i) Find the length of the interval AB **1**

(ii) Find the gradient of the interval AB **1**

(iii) Show that the equation of the line AB is $x+2y-7=0$ **1**

(iv) Determine the size of the acute angle at which AB meets the y axis **1**

(v) State the equation of the line through C which is parallel to AB. **2**

(vi) Find the exact distance between these two parallel lines **2**

(vii) Calculate the exact area of the triangle ABC **1**

b Find the vertex, focus and directrix of the parabola $(x-3)^2=8(y+1)$ **3**

Question 3 (12 Marks) Use a SEPARATE writing booklet.

a Differentiate with respect to x

(i) $(1+x^2)^6$ **2**

(ii) $\dfrac{e^x + e^{3x}}{e^{2x}}$ **2**

b (i) Find $\int \sec^2 7x\,dx$ **2**

(ii) Evaluate $\int_1^3 \dfrac{x}{1+x^2}dx$ **2**

c A plane flew 250 km on a bearing of 070^0 and then 100 km due east. Find its distance and bearing from its starting point. **2, 2**

Question 4 (12 Marks) Use a SEPARATE writing booklet.

a State the domain and range of the function $y = \dfrac{3}{\sqrt{x-2}}$ **2**

b Make a sketch of a right square pyramid where the base is a square of side 8 cm and the perpendicular height is 6 cm. Find the total surface area of the pyramid. **1, 3**

c Solve the equation $2x^2 + 6x + 3 = 0$. Leave your answer in exact form. **1**

d Find the values of k for which the equation $x^2 + kx + k + 3 = 0$ has

(i) a root of 5 **1**

(ii) equal roots **2**

e Express $18^4 - 15^4$ as a product of prime factors **2**

Question 5 (12 Marks) Use a SEPARATE writing booklet.

a Find the common ratio of a geometric progression with first term 1 if its limiting sum is $\sec^2\theta$ **2**

b By walking 200m directly towards a tower, an observer notes that the angle of elevation of its top has increased from 20^0 to 40^0. How high is the tower? **3**

c Find the coordinates of the stationary points on the curve $y = 2x^3 - 9x^2 + 12x$ and determine their nature. **4**

d Prove that $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$ **3**

Question 6 (12 Marks) Use a SEPARATE writing booklet.

a Find $\lim_{x \to 3} \frac{x^3 - 27}{x - 3}$ **2**

b A bag contains 100 counters numbered from 1 to 100. One counter is drawn at random. Find the probability that the number is
(i) a multiple of 6 or 7 **1**
(ii) a multiple of 6 or 7 but not both **1**

c Solve the equation $2(2x+3)^2 - 3(2x+3) = 5$ **2**

d (i) A rail line is to be built between towns A and B. Starting from town A, a team lays 10 km of track on the first day, 12 km the next, then 14 km and so on. Show that the total length of track laid by the end of the n th day is $n^2 + 9n$ km. **1**

(ii) Another team starts from town B, laying 4 km on the first day, 8 km the next, then 12 km and so on. Find the total length of track laid by the end of the n th day. **2**

(iii) Given that when the teams meet, the team from B have laid 18 km more track than the other team, find how many days the job took and the distance of A from B. **2, 1**

Question 7 (12 Marks) Use a SEPARATE writing booklet.

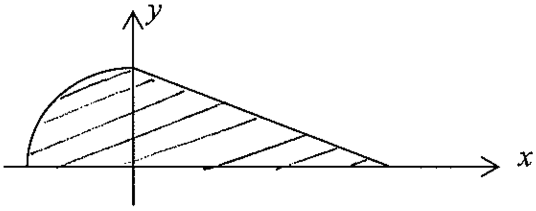
a Evaluate $\sum_{r=1}^{3} r^{r-1}$ **1**

b A particle moves along a straight line so that its displacement, x metres, from a fixed point O is given by $x = t^3 - 6t^2 + 9t + 5$, where t is measured in seconds.
(i) What is the initial displacement of the particle? **1**
(ii) When and where is the particle at rest? **3**
(iii) What is the average velocity of the particle over the first 3 seconds? **1**
(iv) What is the average speed of the particle over the first 3 seconds? **1**

c Sir Richard Brampton has introduced a new VERGIN Bank savings account where interest is compounded continuously so that the principal P at any time is given by $P = Be^{kt}$ where B and k are constants and time t is measured in days. Advertisements state that while other banks offer a rate of 5% pa, VERGIN Savings deposits will grow by 10% in just 700 days.
(i) Show that $\frac{dP}{dt} = kP$ **1**
(ii) Find the value of k correct to 5 significant figures **2**
(iii) How long (to the nearest day) does it take for the principal to increase by 2%? **2**

Question 8 (12 Marks) Use a SEPARATE writing booklet.

a



The shaded area in the diagram is bounded in the second quadrant by part of the circle $x^2 + y^2 = 9$, and in the first quadrant by the straight line $x + 2y - 6 = 0$. Use calculus to find the volume of the solid generated by rotating the shaded area around the x axis. **4**

b Find the derivative of

(i) $\ln x^x$ **1**

(ii) $\ln(\ln x^x)$ **1**

c Solve $e^{\ln x} = \log_3 9$ **2**

d The volume of a closed cylinder of radius r cm is 250π cm^3.

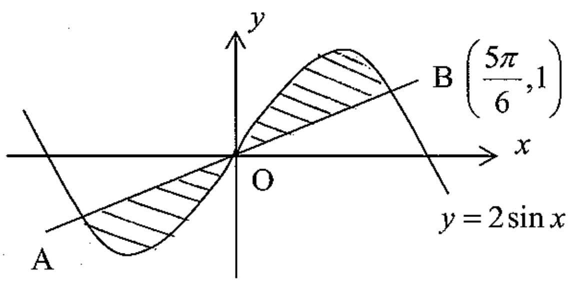
(i) Show that the total surface area is given by $S = 2\pi r^2 + \dfrac{500\pi}{r}$ **1**

(ii) Find the minimum surface area. **3**

Question 9 (12 Marks) Use a SEPARATE writing booklet.

a Find the centre and radius of the circle $x^2 + y^2 - 4x + 6y + 10 = 0$ **2**

b



(i) State the equation of the line AB **1**

(ii) Find the exact value of the shaded area. **3**

c (i) State the coordinates of the two points where the curve $y = -(x^2 + 1)(x^2 - 4)$ crosses the x axis. **1**

(ii) Find the area bounded by the curve and the x axis. **2**

(iii) Use Simpson's Rule with 5 function values to obtain an approximation to this area. **3**

Question 10 (12 Marks) Use a SEPARATE writing booklet.

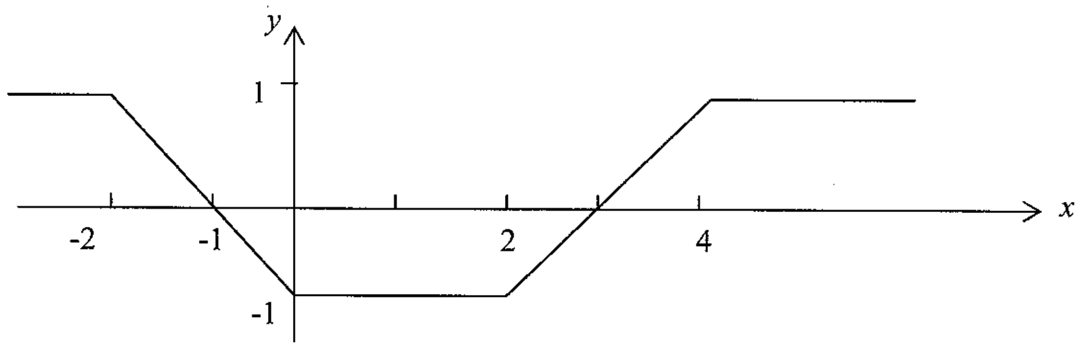
a $\$A_n$ represents the amount owing at the end of n months when an amount of $\$A_0$ is borrowed, monthly repayments of $\$P$ are made, and $r = 1 + \frac{R}{100}$ where $R\%$ is the monthly rate of interest charged.

(i) SHOW that $A_3 = A_0 r^3 - P(1 + r + r^2)$ **2**

(ii) Find the amount owing at the end of 1 year if monthly repayments of \$800 are made on a loan of \$50000 with interest at 9% pa. **3**

(iii) If the loan is to be completely paid off at the end of a further 4 years, what should be the new value of the regular monthly payment $\$P$? **3**

b



Copy the diagram of $y = f'(x)$ onto the MIDDLE OF A NEW PAGE in your writing booklet. Given that $f(0) = 0$, sketch the curve $y = f(x)$ on the SAME AXES. **4**

END OF PAPER

(TSC) 2U Trial Aug 05

1/ a $60 \text{ km/h} = 1 \text{ km/min} = \frac{1000}{60} \text{ m/s} = \frac{50}{3} \text{ m/s} = \underline{16.67 \text{ m/s}}$ (2)

b If $x = 0.31515\dot{1}\dot{5}$ $\therefore 0.3\dot{1}\dot{5} = \frac{312}{990} = \underline{\frac{52}{165}}$ (2)

$100x = 31.51515\dot{1}\dot{5}$

$99x = 31.2$

c $\frac{x-1}{3} - \frac{x-2}{5} = 3$

$5(x-1) - 3(x-2) = 45$

$2x + 1 = 45$ (2)

$\underline{x = 22}$

d $\frac{1}{2\sqrt{3}-1} + \frac{1}{2\sqrt{3}+1} = \frac{(2\sqrt{3}+1)+(2\sqrt{3}-1)}{(2\sqrt{3}-1)(2\sqrt{3}+1)} = \underline{\frac{4\sqrt{3}}{11}}$ (2)

e 4% pa $\equiv \frac{1}{3}\%$ pm, Final value is $\$5000\left(1+\frac{1}{300}\right)^{36} = \underline{\$5636.36}$ (2)

f If $|x+3| < 4$ then $-4 < x+3 < 4$ (2)

$\underline{-7 < x < 1}$

2/ $A(-1,4)$, $B(3,2)$, $C(2,5)$ (1)

a. $AB^2 = (3+1)^2 + (2-4)^2 = 16+4$ so $\underline{AB = 2\sqrt{5}}$

(ii) Grad AB $= \frac{2-4}{3-(-1)} = \underline{-\frac{1}{2}}$ (1)

(iii) Eqn of AB $y - 4 = -\frac{1}{2}(x+1)$

$2y - 8 = -x - 1$ (1)

$\underline{x + 2y - 7 = 0}$

(iv) $\theta = \tan^{-1}|\text{grad } AB| = \tan^{-1}\frac{1}{2} = 26°34'$ with x-axis (1)

$\therefore$ AB meets y axis at $\underline{63° 26'}$

(v) If // AB eqn is $x + 2y + c = 0$ and $c = -12$ (2)

ie thro C so $2 + 10 + c = 0$

$\underline{\text{Eqn is } x + 2y - 12 = 0}$

(vi) Dist of C from AB is $\frac{|2 + 2\times5 - 7|}{\sqrt{1^2+2^2}} = \frac{5}{\sqrt{5}}$ (2)

Dist between lines is $\underline{\sqrt{5} \text{ units}}$

(vii) Area $\Delta = \frac{1}{2} AB \times \perp \text{dist} = \frac{1}{2}(2\sqrt{5})(\sqrt{5}) = \underline{5 \text{ u}^2}$ (1)

b Parabola $(x-3)^2 = 8(y+1)$

Focal length $a = 2$, $\underline{\text{Vertex } (3,-1)}$

$\underline{\text{Focus } (3,1)}$ (3)

$\underline{\text{Directrix } y = -3}$

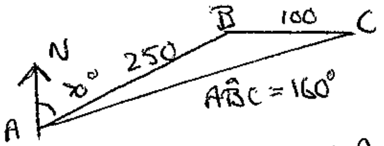
3/ a (i) $\frac{d}{dx}(1+x^2)^6 = 6(1+x^2)^5 2x = \underline{12x(1+x^2)^5}$ (2)

(ii) $\frac{d}{dx}\ \frac{e^x + e^{3x}}{e^{2x}} = \frac{d}{dx}(e^{-x} + e^x) = \underline{e^x - e^{-x}}$ (2)

b (i) $\int \sec^2 7x\,dx = \underline{\frac{1}{7}\tan 7x + c}$ (2)

(ii) $\int_1^3 \frac{x}{1+x^2}\,dx = \frac{1}{2}\left[\ln(1+x^2)\right]_1^3 = \frac{1}{2}(\ln 10 - \ln 2) = \underline{\frac{1}{2}\ln 5}$ (2)

c



By Cos Rule

$AC^2 = 250^2 + 100^2 - 2\times250\times100\cos160^\circ$

$AC = \sqrt{119484.6} \doteqdot 345.7$

By Sin Rule $\frac{\sin A}{100} = \frac{\sin 160^\circ}{345.7}$

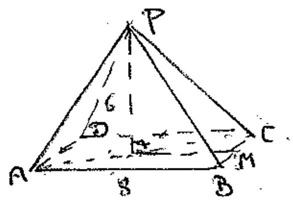
$\therefore A = \sin^{-1} 0.0989 = 5^\circ 41'$ (2,2)

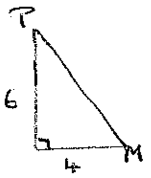
Plane is 345.7 km on a bearing of 075° 41' from starting point

(2)

4/ a $y = \frac{3}{\sqrt{x-2}}$ has domain $x > 2$ and range $y > 0$

b





By Pythag $PM^2 = 6^2 + 4^2 = 52$

Area $\Delta PBC = \frac{1}{2}\times 8\times\sqrt{52} = 8\sqrt{13}$ (1,3)

$\therefore$ TSA $= \underline{64 + 32\sqrt{13}\ \text{cm}^2} \doteqdot 179.4\text{cm}^2$

(1)

c If $2x^2 + 6x + 3 = 0$, $x = \frac{-6 \pm \sqrt{36-24}}{4} = \underline{\frac{-3\pm\sqrt{3}}{2}}$

(1)

d Eqn $x^2 + kx + k + 3 = 0$

(i) Root of 5 $\quad 25 + 5k + k + 3 = 0$ so $6k = -28$ and $\underline{k = -\frac{14}{3}}$ (2)

(ii) Equal roots $\quad \Delta = k^2 - 4(k+3) = 0 = k^2 - 4k - 12$

$= (k-6)(k+2)$ so $\underline{k = 6 \text{ or } -2}$

e $18^4 - 15^4 = 3^4(6^4 - 5^4) = 3^4(6^2-5^2)(6^2+5^2)$ (2)

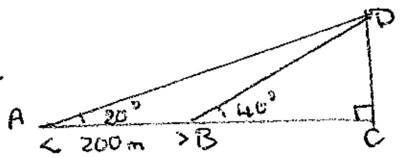
$= \underline{3^4 \times 11 \times 61}$

(2)

5/ a GP $a = 1$, $S = \frac{a}{1-r} = \sec^2\theta$, common ratio $\underline{r = 1 - \cos^2\theta = \sin^2\theta}$

$\therefore 1 - r = \cos^2\theta$,

b



$A\hat{D}B = 20^\circ$ so ΔABD is isos

$\therefore DC = 200\sin 40^\circ$ (3)

Tower is 128.6 m high

c If $y = 2x^3 - 9x^2 + 12x$, $\frac{dy}{dx} = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$

$\frac{d^2y}{dx^2} = 12x - 18$, when $x = 1$, $y = 5$, $\frac{d^2y}{dx^2} < 0$ so local max at (1,5) (4)

" $x = 2$, $y = 4$, $\frac{d^2y}{dx^2} > 0$ " " min at (2,4)

d $\frac{\sin\theta}{1+\cos\theta} = \frac{\sin\theta}{1+\cos\theta} \times \frac{1-\cos\theta}{1-\cos\theta} = \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta} = \frac{1-\cos\theta}{\sin\theta}$ (3)

6/ a $\lim_{x\to 3} \frac{x^3-27}{x-3} = \lim_{x\to 3} (x^2+3x+9) = 27$ (2)

b Draw 1 counter from bag of counters no. 1 – 100

(i) P(mult of 6 or 7) = P(mult 6) + P(mult 7) − P(mult of both 6 and 7) (1)

$= \frac{16}{100} + \frac{14}{100} - \frac{2}{100} = \frac{28}{100} = \frac{7}{25}$

(ii) P(mult of 6 or 7 but not both) = P(mult of 6 or 7) − P(mult of both) (1)

$= \frac{28}{100} - \frac{2}{100} = \frac{26}{100} = \frac{13}{50}$

c $2(2x+3)^2 - 3(2x+3) = 5$ (2)

$[2(2x+3) - 5][(2x+3) + 1] = 0$ $\therefore$ $x = -\frac{1}{4}$ or -2

$[4x+1][2x+4] = 0$

d From A: $10 + 12 + 14 + \cdots$ AP $a = 10$, $d = 2$ (1)

(i) $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[20 + 2(n-1)] = n(n+9) = n^2 + 9n$

(ii) From B: $4 + 8 + 12 + \cdots$ AP $a = d = 4$ (2)

$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[8 + 4(n-1)] = 2n(n+1) = 2n^2 + 2n$

(iii) If $2n^2 + 2n = n^2 + 9n + 18$ so (positive) $n = 9$

$n^2 - 7n - 18 = 0 = (n-9)(n+2)$ (2,1)

Total length is $2(81+81) + 18 = 324 + 18 = 342$

In 9 days a ~~total~~ of 342 km of track laid (1)

7/ a $\sum_{r=1}^{3} r^{r-1} = 1^0 + 2^1 + 3^2 = 1 + 2 + 9 = 12$

b $x = t^3 - 6t^2 + 9t + 5$

$\dot{x} = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t-1)(t-3)$ (1)

$\ddot{x} = 6t - 12 = 6(t-2)$

(i) When $t = 0$, $x = 5$ initial displ is 5 (3)

(ii) $\dot{x} = 0$ when $t = 1$ or 3 when $t = 1$, $x = 9$ (1)

" $t = 3$, $x = 5$ (1)

(iii) From $t = 0$ to $t = 3$ no change in displ so av. vel = 0

(iv) " " " " dist travelled is 4+4 so average speed is $\frac{8}{3}$ m/s

c Given $P = Be^{kt}$ (B & k const) (1)

(i) $\frac{dP}{dt} = Bke^{kt} = k(Be^{kt}) = kP$

(ii) Given $P = 1.1B$ when $t = 700$ (2)

$1.1 = e^{700k}$ so $k = \frac{\ln 1.1}{700} \doteqdot 0.00013616$ to 5 sf

$\ln 1.1 = 700k$

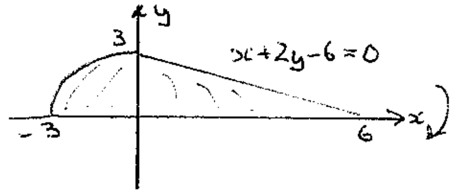
(iii) If $P = 1.02B$

$1.02 = e^{kt}$

$t = \frac{\ln 1.02}{k} \doteqdot 145.44$ (2)

Takes approx 145 days to increase by 2%

8/ a



$\delta V \doteqdot \pi r^2 h = \pi y^2 \delta x$

Vol $V = \pi\int_{-3}^{0}(9-x^2)\,dx + \pi\int_0^6 \left(\frac{6-x}{2}\right)^2 dx$

$= \pi\left[9x - \frac{x^3}{3}\right]_{-3}^{0} + \frac{\pi}{4}\left[36x - 6x^2 + \frac{x^3}{3}\right]_0^6$

$= 18\pi + \frac{\pi}{4}(72) = \underline{36\pi \ u^3}$ (4)

b (i) $\frac{d}{dx}\ln x^x = \frac{d}{dx}x\ln x = x\left(\frac{1}{x}\right) + 1.\ln x = \underline{1+\ln x}$ (1)

(ii) $\frac{d}{dx}\ln(\ln x^x) = \underline{\frac{1+\ln x}{x\ln x}}$ (1)

c $e^{\ln x} = \log_3 9 = \log_3 3^2 = 2\log_3 3$

$\underline{x = 2}$ (2)

d Cyl vol $V = \pi r^2 h = 250\pi$ so $h = \frac{250}{r^2}$ (1)

(i) TSA $A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r\left(\frac{250}{r^2}\right) = 2\pi r^2 + \frac{500\pi}{r}$

(ii) $\frac{dA}{dr} = 4\pi r - 500\pi r^{-2} = \frac{4\pi}{r^2}(r^3 - 125)$

$\frac{d^2A}{dr^2} = 4\pi + 1000\pi r^{-3}$

St. Pt when $\frac{dA}{dr} = 0$ so $r = 5$, $\frac{d^2A}{dr^2} > 0$ so minimum $A = 2\pi.25 + \frac{500\pi}{5}$ (3)

Minimum surface area is 150π cm^2

9/ a $x^2 + y^2 - 4x + 6y + 10 = 0$ (2)

$(x-2)^2 + (y+3)^2 = 3$ Centre $(2,-3)$, radius $\sqrt{3}$

b (i) OB has gradient $1/\frac{5\pi}{6} = \frac{6}{5\pi}$ so eqn of AB is $\underline{y = \frac{6x}{5\pi}}$ (1)

or $\underline{6x - 5\pi y = 0}$

(ii) Two regions have same area

Shaded area $A = 2\int (y_{upper} - y_{lower})\,dx$

$= 2\int_0^{\frac{5\pi}{6}}\left(2\sin x - \frac{6x}{5\pi}\right)dx$

$= 2\left[-2\cos x - \frac{3x^2}{5\pi}\right]_0^{\frac{5\pi}{6}}$

$= 2\left\{\left(\sqrt{3} - \frac{3.5\pi}{36}\right) - (-2)\right\}$ (3)

$= \underline{4 + 2\sqrt{3} - \frac{5\pi}{12} \ u^2}$ (1)

c (i) $y = -(x^2+1)(x^2-4) = -(x^2+1)(x-2)(x+2)$ crosses x-axis at $\underline{(-2,0), (2,0)}$

(ii) Area $= \int_{-2}^{2} y\,dx = 2\int_0^2 -(x^4 - 3x^2 - 4)\,dx$ (2)

$= 2\left[-\frac{x^5}{5} + x^3 + 4x\right]_0^2 = 2\left(16 - \frac{32}{5}\right) = \underline{19\frac{1}{5} \ u^2}$

(iii) By Simpson's Rule with 5 function values $h = \frac{4}{4} = 1$

Area $\doteqdot \frac{1}{3}\left[f(-2) + 4f(-1) + 2f(0) + 4f(1) + f(2)\right]$

$\approx \frac{1}{3}\left[0 + 4\times 6 + 2\times 4 + 4\times 6 + 0\right]$

$\approx \frac{56}{3} = \underline{18\frac{2}{3} \ u^2}$ (3)

10/ a (i) $\$A_n$ owing after n periods, monthly payment $\$P$, $R\%$ pm and $r = 1 + \frac{R}{100}$

$A_1 = A_0 r - P$

$A_2 = A_1 r - P = (A_0 r - P)r - P = A_0 r^2 - P(1+r)$

$A_3 = A_2 r - P = A_0 r^3 - P(1+r)r - P = \underline{A_0 r^3 - P(1+r+r^2)}$ (2)

(ii) $P = 800$, $n = 12$, $R = \frac{9}{12}$ so $r = 1 + \frac{3}{400} = 1.0075$, $A_0 = 50000$

$A_{12} = 50\,000 \times 1.0075^{12} - P(1 + r + \cdots + r^{11})$

$= 50\,000 \times 1.0075^{12} - P\frac{r^{12}-1}{r-1}$

$= 50\,000 \times 1.0075^{12} - 800\frac{1.0075^{12}-1}{0.0075}$ (3)

$= 44\,684.28$ — After 1 year, owe \$44 684.28

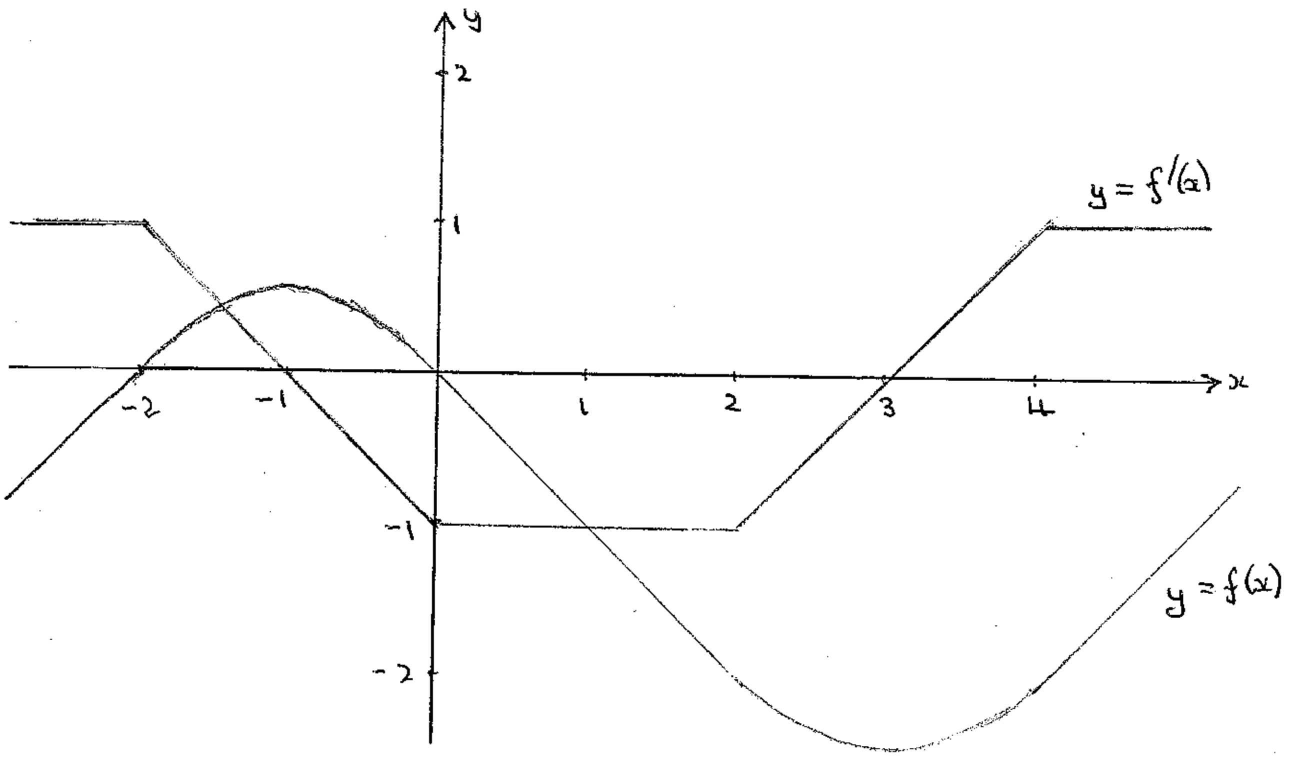
(iii) Now take $A_0 = 44\,684.28$, $n = 48$ and find P if $A_{48} = 0$

$0 = A_0 r^{48} - P\frac{r^{48}-1}{r-1}$ (3)

so $P = \frac{A_0 r^{48}(r-1)}{r^{48}-1}$ — Payment needs to be \$1111.97

$\doteqdot 1111.968$

b



(4)